Blad 1/2 Kansrekening

1 Teler A levert 4x zoveel als teler B

$P(A \text{ rot}) = \frac{1}{10}$

$P(B \text{ rot}) = \frac{3}{10}$

$$P(\text{Bij A vandaan} \mid \text{Appel rot}) = \frac{P(\text{Bij A vandaan} \cap \text{Appel rot})}{P(\text{Appel rot})}$$

$$= \frac{P(\text{Appel rot} \mid \text{Bij A vandaan}) \cdot P(\text{Bij A vandaan})}{P(\text{Appel rot})}$$

$$P(\text{Appel rot}) = \frac{4}{5} \cdot \frac{1}{10} + \frac{1}{5} \cdot \frac{3}{10} = \frac{7}{50}$$

$$P(\text{Bij A vandaan}) = \frac{4}{5}$$

$$P(\text{Appel rot} \mid \text{Bij A vandaan}) = \frac{1}{10}$$

Deze waarden invullen geeft:

$$P(\text{Bij A vandaan} \mid \text{Appel rot}) = \frac{\frac{1}{10} \cdot \frac{4}{5}}{\frac{7}{50}} = \frac{\frac{4}{50}}{\frac{7}{50}} = \frac{4}{7}$$

60

2 4 8 16 32 64

2a $Bin(5, \frac{1}{2})$ $\quad X \sim Bin(5, \frac{1}{2}) \Rightarrow n = 5,\ p = \frac{1}{2}$

$$f_X(k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{5}{k}\left(\frac{1}{2}\right)^k \left(1-\frac{1}{2}\right)^{5-k}$$

$$= \binom{5}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{5-k} = \binom{5}{k}\left(\frac{1}{2}\right)^5 = \binom{5}{k}\frac{1}{2^5} = \frac{1}{32}\binom{5}{k}$$

$$F_X(k) = \sum_{k=0}^{5} \frac{1}{32}\binom{5}{k} = \frac{120}{32}\sum_{k=0}^{5}\frac{1}{k!(5-k)!} = \frac{15}{4}\sum_{k=0}^{5}\frac{1}{k!(5-k)!}$$

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$\mathbb{E}[X] = \sum_{k=0}^{5} k\binom{5}{k}\frac{1}{2^5} = \sum_{k=1}^{5} k\binom{5}{k}\frac{1}{2^5} = \frac{1}{2^5}\sum_{k=1}^{5} k\binom{5}{k}$$

$$= \frac{1}{32}\left[1\cdot\binom{5}{1} + 2\cdot\binom{5}{2} + 3\cdot\binom{5}{3} + 4\binom{5}{4} + 5\cdot\binom{5}{5}\right]$$

$$= \frac{1}{32}\left[1\cdot 5 + 2\cdot 10 + 3\cdot 10 + 4\cdot 5 + 5\cdot 1\right]$$

3

$$= \frac{1}{32}(5+20+30+20+5) = \frac{1}{32}\cdot 80 = \frac{80}{32} = \frac{40}{16} = \frac{10}{4} = 2.5$$

b $f(x) = C(30 + x^5)$ voor $0 \le x \le 2$

$f(x) = 0$ elders

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} C(30+x^5)\,dx + \int_{2}^{\infty} 0\,dx$$

$$= \int_0^2 C(30+x^5)\,dx = \left[C\cdot 30x + \frac{1}{6}C\cdot x^6\right]\Big|_0^2$$

5

$$= 60C + \frac{64}{6}C = 60C + \frac{32}{3}C = \frac{212}{3}C = 1$$

$$\Rightarrow C = \frac{3}{212}$$

8

2c De wet van de grote aantallen:

$$\lim_{n\to\infty} (\bar{X}_n - \mathbb{E}[\bar{X}_n]) = 0$$

Centrale limietstelling:

voor $n \to \infty$ $\quad \sqrt{n}\,\frac{\bar{X}_n - \mathbb{E}[\bar{X}_n]}{\sigma} = \frac{\bar{X}_n - \mathbb{E}[\bar{X}_n]}{\sqrt{Var(\bar{X}_n)}} \to 0$

- En voorbeeld waarvoor geldt dat:

$$Var(\bar{X}_n) \leq (\bar{X}_n - \mathbb{E}[\bar{X}_n])^2 = \bar{X}_n^2 - 2\bar{X}_n \mathbb{E}[\bar{X}_n] + (\mathbb{E}[\bar{X}_n])^2$$

voor $n \to \infty$

$$\sqrt{n}\,\frac{\bar{X}_n - \mu}{\sigma}$$

$$P(\bar{X}_n - \mu \geq a) \leq \frac{1}{a^2} Var(\bar{X}_n)$$

0

$$\mathbb{E}[\bar{X}_n^2] - (\mathbb{E}[\bar{X}_n])^2 \leq \bar{X}_n^2 - 2\bar{X}_n \mathbb{E}[\bar{X}_n] + (\mathbb{E}[\bar{X}_n])^2$$

$$\bar{X}_n^2 - 2\bar{X}_n \mathbb{E}[\bar{X}_n] + 2(\mathbb{E}[\bar{X}_n])^2 - \mathbb{E}[\bar{X}_n^2] \geq 0$$

$$\overset{(1)}{\Rightarrow} (\mathbb{E}[\bar{X}_n])^2 - \cancel{2(\mathbb{E}[\bar{X}_n])^2} + \cancel{2(\mathbb{E}[\bar{X}_n])^2} - \mathbb{E}[\bar{X}_n^2] \geq 0$$

$$\Rightarrow (\mathbb{E}[\bar{X}_n])^2 - \mathbb{E}[\bar{X}_n^2] \geq 0 \quad \Rightarrow \quad Var(\bar{X}_n) \leq 0$$

(1) Geldt, omdat het aant de wet van de grote aantallen moet voldoen

3 $X \sim U(-2,2) \Rightarrow \alpha = -2, \beta = 2$

$f_X(x) = \frac{1}{\beta - \alpha} = \frac{1}{2--2} = \frac{1}{4}$

Controle: $F_X(x) = \int f_X(x)\,dx = \int \frac{1}{4}\,dx = \frac{1}{4}x$

en $\int_{-2}^{2} f_X(x)\,dx = \frac{1}{4}x\Big|_{-2}^{2} = \frac{1}{4}(2--2) = 1$

6/10

$Y = X^2$

$Cov(X,Y) = Cov(X,X^2) = \mathbb{E}[X \cdot X^2] - \mathbb{E}[X]\mathbb{E}[X^2] = 0$

- want de functie is oneven $\Rightarrow \mathbb{E}[X^3] = \mathbb{E}[X] = 0$

$F_X(x) = \frac{1}{4}x$

~~Dus X en Y zijn ongecorreleerd~~. onafhankelijk en daarom ~~ook ongecorreleerd (stelling)~~

- $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] \Rightarrow Var(X+Y) = Var(X) + Var(Y)$

$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\mathbb{E}[X] = \int_{-2}^{2} x f_X(x)\,dx = \int_{-2}^{2} \frac{1}{4}x\,dx = \frac{1}{8}x^2\Big|_{-2}^{2} = \frac{1}{8}2^2 - \frac{1}{8}(-2)^2 = 0$ *

$\mathbb{E}[X^2] = \int_{-2}^{2} x^2 f_X(x)\,dx = \int_{-2}^{2} \frac{1}{4}x^2\,dx = \frac{1}{12}x^3\Big|_{-2}^{2} = \frac{1}{12}2^3 - \frac{1}{12}(-2)^3 =$

$= \frac{1}{12}\cdot 8 + \frac{1}{12}\cdot 8 = \frac{16}{12} = \frac{4}{3}$

$Var(X) = \frac{4}{3}$ $\left(Var(X) = \frac{(\beta-\alpha)^2}{12} = \frac{(2--2)^2}{12} = \frac{4}{3}\right)$

$$\mathrm{Var}(Y) = \mathrm{Var}(X^2) = \mathbb{E}[X^4] - (\mathbb{E}[X^2])^2$$

$$\mathbb{E}[X^4] = \int_{-2}^{2} x^4 f_X(x)\,dx = \int_{-2}^{2} \tfrac{1}{4} x^4\,dx = \tfrac{1}{20} x^5 \Big|_{-2}^{2} = \tfrac{1}{20} 2^5 - \tfrac{1}{20}(-2)^5$$

$$= \frac{32}{20} + \frac{32}{20} = \frac{64}{20} = \frac{16}{5}$$

$$\mathrm{Var}(Y) = \frac{16}{5} - \left(\frac{4}{3}\right)^2 = \frac{16}{5} - \frac{16}{9} = \frac{64}{45}$$

R

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X+X^2) = \mathbb{E}[(X+X^2)^2] - (\mathbb{E}[X+X^2])^2$$

$$= \mathbb{E}[X^2 + 2X^3 + X^4] - (\mathbb{E}[\cancel{X^4} + \cancel{2}X^2 + \cancel{X^4}])^2$$

$$= \cancel{\mathbb{E}[X^2] + 2\mathbb{E}[X^3] + \mathbb{E}[X^4] - \mathbb{E}[X^2] - 2\mathbb{E}[X^3] - \mathbb{E}[X^4]}$$

$$= \mathbb{E}[X^2] + 2\mathbb{E}[X^3] + \mathbb{E}[X^4] - (\mathbb{E}[X] + \mathbb{E}[X^2])^2$$

$$= \mathbb{E}[X^2] + 2\mathbb{E}[X^3] + \mathbb{E}[X^4] - (\mathbb{E}[X])^2 - 2\mathbb{E}[X]\mathbb{E}[X^2] - (\mathbb{E}[X^2])^2$$

$$= \frac{4}{3} + 2\cdot 0 + \frac{16}{5} - 0^2 - 2\cdot 0\cdot\frac{4}{3} = \frac{4}{3} + \frac{16}{5} = \frac{68}{15}$$

Klopt niet ergen een foutje gemaakt

4 $P(\text{partij 1 wint}) = P(\text{partij 2 wint}) = \frac{1}{2}$

7 $\cancel{\mathbb{E}(\text{levensduur winnaar}) = \sum_{n=0}^{\infty} 4n\left(\frac{1}{2}\right)^n + 4}$

$\cancel{= 4 + \sum_{n=1}^{\infty} (2^2)^n \frac{1}{2^n} = 4 + \sum_{n=1}^{\infty} \frac{2^{2n}}{2^n} = 4 + \sum_{n=1}^{\infty} 2^n}$

$\cancel{= 4 + \sum_{n=1}^{\infty} \frac{4n}{2^n} = 4 + 4\sum_{n=1}^{\infty} \frac{1}{2^n} = 4 + 4 \cdot 1 = 8 \text{ jaar}}$

- $P(X \leq q_{0.5}) = 0.5$ X is de levensduur van de winnaar

  $q_{0.5}$ is dan de mediaan

$F_X(x) = \cancel{\int f_X(x)\,dx = \int \left(\frac{1}{2}\right)^x dx = \int 2^{-x} dx}$

$= \sum_{x'=0}^{x} f_X(x') = \sum_{x'=0}^{x} \left(\frac{1}{2}\right)^{x'}$

Voor $q_{0.5} = 6$ is $P(F_X(x) \leq q_{0.5}) = 0.5$

$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\cancel{\mathbb{E}[X^2] = \sum_{n=0}^{\infty} 4^2 \left(\frac{1}{2}\right)^n + 4 = 4 + 16\sum_{n=0}^{\infty} \frac{1}{2^n}}$

$\cancel{Var(X) = 16 - 8^2}$

$\mathbb{E}[X] = \sum_{n=0}^{\infty} 4n\left(\frac{1}{2}\right)^n + 4 = 4 + \sum_{n=0}^{\infty} \frac{4n}{2^n} = 4 + \sum_{n=1}^{\infty} \frac{4n}{2^n} = 4 + 8 = 12$ jaar

$\mathbb{E}[X^2] = \sum_{n=0}^{\infty} (4n+4)^2 \left(\frac{1}{2}\right)^n = \sum (16n^2 + 32n + 16)\left(\frac{1}{2}\right)^n = 16 + 96 + 96 = 208$ jaar

nee

$Var(X) = 208 - 12^2 = 208 - 144 = 64$ jaar

~~2c~~ $\cancel{\lim_{n\to\infty} E[\bar{X}_n - \mu] = 0}$

$$\cancel{\lim_{n\to\infty} (\bar{X}_n - E[\bar{X}_n]) = 0}$$

$$\cancel{\sqrt{n}\,\frac{\bar{X}_n - E[\bar{X}_n]}{\sigma}}$$

5a 13 Schoppenkaarten

2 kaarten trekken aanname: zonder teruglegging

Er zijn 5 even schoppens, nl. 2, 4, 6, 8 en 10.

$$P(2 \text{ even schoppens}) = \frac{5}{13} \cdot \frac{4}{12}$$

b X = aantal gescoorde doelpunten van team 1

Y = aantal gescoorde doelpunten van team 2

$X \sim \text{Pois}(\mu_X) = \text{Pois}(2)$; $\mu_X = 2$

$Y \sim \text{Pois}(\mu_Y) = \text{Pois}(2)$; $\mu_Y = 2$

$$P(X=Y) = \sum_{i=0}^{\infty} P(X=i \cap Y=i) \overset{(1)}{=} \sum_{i=0}^{\infty} P(X=i)P(Y=i)$$

$$= \sum_{i=0}^{\infty} \left(\frac{\mu^i}{i!} e^{-\mu}\right)^2 = \sum_{i=0}^{\infty} \frac{\mu^{2i}}{(i!)^2} e^{-2\mu}$$

$$= \sum_{i=0}^{\infty} \frac{2^{2i}}{(i!)^2} e^{-2\cdot 2} = \sum_{i=0}^{\infty} \frac{4^i}{(i!)^2} e^{-4}$$

(1) Vanwege onafhankelijkheid uit aanname.